Math 522 Exam 10 Solutions

1. Prove that $\sum_{r|n} \frac{\mu(r)}{d(r)} = 2^{-s}$, where s is the number of different primes dividing n, i.e. $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$.

   Set $G(n) = \sum_{r|n} \frac{\mu(r)}{d(r)}$. Note that $\frac{\mu}{d}$ is multiplicative, since both the numerator and denominator are (and the denominator is never zero). Because $G = \frac{\mu}{d} \star 1$, G is also multiplicative. Hence $G(n) = \prod_{i=1}^{s} G(p_i^{a_i}) = \prod_{i=1}^{s} \left(\frac{\mu(1)}{d(1)} + \frac{\mu(p_i)}{d(p_i)} + \frac{\mu(p_i^2)}{d(p_i^2)} + \cdots \right) = \prod_{i=1}^{s} \left(\frac{1}{1} + \frac{-1}{2} + 0 + \cdots \right) = \prod_{i=1}^{s} \left(\frac{1}{2} \right) = 2^{-s}$.

   The number of different primes dividing n is called $\omega(n)$, which is interesting in its own right. This is an additive function (not multiplicative); however exponentiating an additive function makes a multiplicative function.

2. Prove that $d^{-1} = \mu \star \mu$. Compute $d^{-1}(27)$, either using this fact or recursively.

   We begin with $1 \star 1 = d$, and multiply both sides by $d^{-1} \star \mu \star \mu$ to get $d^{-1} \star \mu \star \mu \star 1 \star 1 = d \star d^{-1} \star \mu \star \mu$. This simplifies to $d^{-1} = \mu \star \mu$, because $d \star d^{-1} = I = 1 \star \mu$ and $\star$ is commutative and associative.

   Using this fact, $(\mu \star \mu)(27) = \sum_{de=27} \mu(d)\mu(e) = \mu(1)\mu(27) + \mu(3)\mu(9) + \mu(9)\mu(3) + \mu(27)\mu(1) = 0$, since $\mu(27) = \mu(9) = 0$.

   Recursively, we calculate $d^{-1}(3), d^{-1}(9), d^{-1}(27)$.
   $d^{-1}(3) = -\sum_{r|3, r<3} d(\frac{3}{r}) d^{-1}(r) = -d(3)d^{-1}(1) = -2$.
   $d^{-1}(9) = -\big(d(9)d^{-1}(1) + d(3)d^{-1}(3)\big) = -(3-4) = 1$.
   $d^{-1}(27) = -\big(d(27)d^{-1}(1) + d(9)d^{-1}(3) + d(3)d^{-1}(9)\big) = -(4-6+2) = 0$.